Quantum Time Scales and the Classical Limit: Analytic Results for Some Simple Systems

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We set us a semiclassical expansion scheme for the quantum dynamics around a time dependent coherent state whose dynamics is (apart from a global phase) completely determined by the underlying classical equations of motion, generalizing [1]. Therefore all next to leading order in this expansion are of essentially quantum character, introducing effects such as linear superposition of states. The rate of increase of quantum corrections as measured by the overlap between zero-th order and exact result can be directly associated to a Lyapunov exponent. In this contribution we explore some simple systems in order to discuss and illustrate some features of the quantum-classical transition.

The semiclassical expansion is a time dependent pertubation where we write the Hamiltonian as $H_q = H_{sc} + \delta$, and by definition we have $\delta = \widehat{H}_q - \widehat{H}_{sc}$.

First we suppose that the classical Hamiltonian can be written as $H_{cls} = \hbar\varpi\alpha^*\alpha + \sum_{m,n} A_{m,n}\left(\alpha^*\right)^m \alpha^n$ where the above variables are related to the canonical ones as $p = \frac{\alpha-\alpha^*}{i\sqrt{\frac{2}{m\varpi\hbar}}}$; $q = \frac{\alpha+\alpha^*}{\sqrt{\frac{2m\varpi}{\hbar}}}$ and by definition we have $A_{1,1} = 0$. The corresponding quantum Hamiltonian is defined as $\widehat{H}_q = \hbar\varpi\widehat{a}^\dagger\widehat{a} + \sum_{m,n} A_{m,n}\left(\widehat{a}^\dagger\right)^m \widehat{a}^n$. Note that $\langle\alpha|\, H_q\, |\alpha\rangle = H_{cls}$. Now the semiclassical Hamiltonian is, by definition

$$\begin{aligned}\widehat{H}_{sc} &= \hbar\varpi\widehat{a}^\dagger\widehat{a} + \sum_{m\neq 0} m\cdot A_{m,m}\left(\alpha^*\right)^{m-1}\alpha^{m-1}(\widehat{a}^\dagger\widehat{a} - \alpha^*\alpha) + \sum_{m,n} A_{m,n}\left(\alpha^*\right)^m\alpha^n \\ &+ \sum_{m\neq n} m\cdot A_{m,n}\left(\alpha^*\right)^{m-1}\alpha^n(\widehat{a}^\dagger - \alpha^*) + \sum_{m,n} n\cdot A_{m,n}\left(\alpha^*\right)^m\alpha^{n-1}(\widehat{a} - \alpha).\end{aligned}$$

Note that $\langle\alpha|\,\widehat{H}_{sc}\,|\alpha\rangle = H_{cls}$. Now the complete evolution can be formally written as

$$|\Psi(t)\rangle = \hat{U}_{sc}(t)\left(1 + \frac{1}{i\hbar}\int_0^t dt_1\Delta_s(t_1) + \frac{1}{(i\hbar)^2}\int_0^t\int_0^{t_1} dt_2dt_2\Delta_s(t_1)\Delta_s(t_2) + ...\right)|\Psi(0)\rangle \tag{1}$$

where $\Delta_s = \hat{U}_{sc}^{\dagger}(t)\delta\hat{U}_{sc}(t)$ and $\hat{U}_{sc}(t) = e^{-\frac{i}{\hbar}\int_0^t \hat{H}_{sc}(t\prime)dt\prime}$.

If we consider only the zeroth order for the equation 1 we have for overlap between two wave functions

$$\langle\psi|\,|\phi\rangle = \langle\psi|\,|\phi\rangle_{sc} + \text{corretions.}$$

Here $|\phi(0)\rangle = |\alpha_2(0)\rangle$ and $|\psi(0)\rangle = |\alpha_1(0)\rangle$, were $|\alpha_1(t)\rangle, |\alpha_2(t)\rangle$ are coherent states then $\langle\psi(t)|\,|\phi(t)\rangle_{sc} = \langle\alpha_1(t)|\,|\alpha_2(t)\rangle$. We know that. [2] $|\langle\alpha_1|\,|\alpha_2\rangle|^2 = \exp(-|\alpha_1 - \alpha_2|^2)$

while the Lyapunov exponent is by definition [3]

$$\lambda = \lim_{t\to\infty}\lim_{\Delta x(0)\to 0}\frac{1}{t}\ln\frac{\Delta x(t)}{\Delta x(0)}. \tag{2}$$

Then we can use the overlap to define a quantum version of the Lyapunov exponent as

$$\lambda = \lim_{t\to\infty}\lim_{\alpha_2(0)\to\alpha_1(0)}\frac{1}{2t}\ln\left[\frac{\ln|\langle\alpha_1(t)|\,|\alpha_2(t)\rangle|^{-1}}{\ln|\langle\alpha_1(0)|\,|\alpha_2(0)\rangle|^{-1}}\right]. \tag{3}$$

This expression can be interpreted as follows. For a regular system the overlap will vary smoothly and because of the term t appearing i the denominator we will have a null exponent, this also means that the leading term of the expansion will survive for a long time compared with the classical chaotic system.

As example we consider the quartic oscillator, it have been object of studies of many auctors as [4, 5], and more recently [6, 7].

The classical Hamiltonian is $H_{cls} = \hbar\varpi\alpha^*\alpha + \lambda\hbar^2(\alpha^*)^2\alpha^2$. and the classical solution is $\alpha(t) = \alpha(0)\exp[(-i\varpi - i2\lambda\hbar|\alpha(0)|^2)t]$.

The quantum Hamiltonian can be written as $\widehat{H}_q = \hbar\varpi\widehat{a}^{\dagger}\widehat{a} + \lambda\hbar^2\left(\widehat{a}^{\dagger}\right)^2\widehat{a}^2$ whose exact solution, for the expectation value for the destruction operator is $\langle\widehat{a}\rangle(t) = \alpha(0)e^{-i\varpi t}e^{-|\alpha(0)|^2(1-\exp(-2it\lambda\hbar))}$.

For the semiclassical analysis, we may write a semiclassical Hamiltonian as $\widehat{H}_{sc} = \hbar\varpi\widehat{a}^{\dagger}\widehat{a} + 2\cdot\lambda\hbar^2\alpha^*\alpha\widehat{a}^{\dagger}\widehat{a} - \lambda\hbar^2\left(\alpha^*\right)^2\alpha^2$. As we have $[\delta, \widehat{H}_{sc}] = 0 \Longrightarrow \delta = \Delta_H = \Delta_s^{\dagger}$.

If we set $|\alpha(0)|^2\hbar = S_{cl}$ we find the usual Ehrenfest time [8, 4], that is $t_e \sim (2\hbar\lambda S_{cl})^{-1/2}$. Although this time says that the average of the operator a is in agreement with the classical evolution, we can not say that we are in a classical regime. As we can see in the figure 1, the Wigner function has quantum correlations that have no classical counterpart .

For a coupled quartic oscillator with the quantum Hamiltonian as

$$\widehat{H}_q = \hbar\varpi(\widehat{a}^{\dagger}\widehat{a} + \widehat{b}^{\dagger}\widehat{b}) + \lambda\hbar^2\left\{\left(\widehat{a}^{\dagger}\right)^2\widehat{a}^2 + (\widehat{b}^{\dagger})^2\widehat{b}^2 + 2\widehat{a}^{\dagger}\widehat{a}\widehat{b}^{\dagger}\widehat{b}\right\} + \mu\hbar(\widehat{a}^{\dagger}\widehat{b} + \widehat{a}\widehat{b}^{\dagger})$$

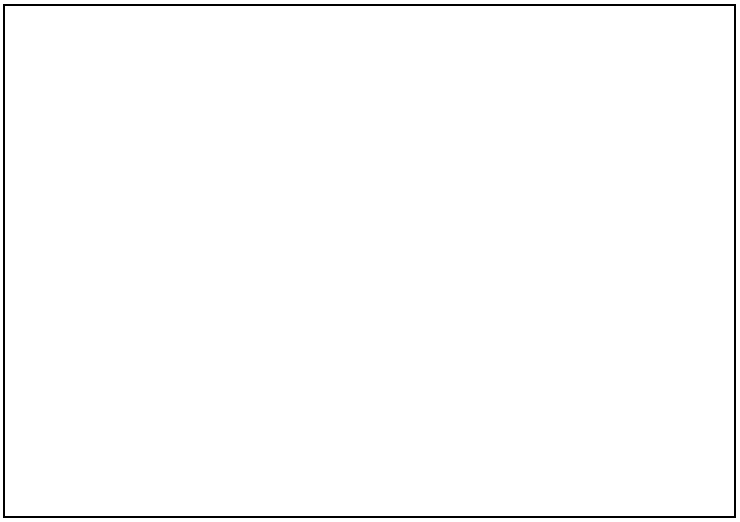


Figure 1: Aproximated Wigner function for short time.

we have as linear entropy $\delta = 1 - Tr[\widehat{\rho_\alpha}^2]$

$$Tr[\widehat{\rho_\alpha}^2] = \sum_{q,n} e^{-2\left|\widehat{\beta_0}\right|^2} \exp(2\widehat{|\alpha_0|}^2 (\sum_{m=0} \frac{(2t\lambda\hbar(n-q))^{2m+2}}{(2m+2)!}) \times \frac{\left|\widehat{\beta_0}\right|^{2n}}{n!} \frac{\left|\widehat{\beta_0}\right|^{2q}}{q!}$$

for a initial state as a product of coherent states and we are looking to one degree of freedom.

If we sum up all $\hbar$ independent terms we have

$$Tr[\widehat{\rho_\alpha}^2] \approx \frac{1}{\sqrt{1 + 16t^2\lambda^2 \left|S^a_{cl} S^b_{cl}\right|}}.$$

In the figure 2 we have the comparison between the exact result and the approximated result, we can see that for short time the $\hbar$ independent terms are the most important ones, as the entanglement possesses a very short ($\hbar$ independent) time scale. This typically quantum time scale is directly related to the linear structure of the Hilbert space, therefore primarily connected to the differences between quantum and classical descriptions which are due to heir different kinematics, rather than dynamics.

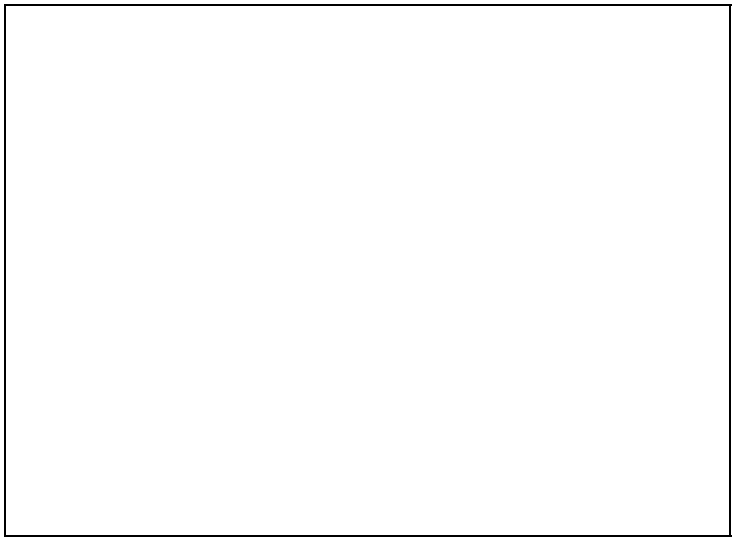


Figure 2: $1-\delta$ for the coupled quartic oscillator. Full line the exact result and dotted for the $\hbar$ independent trace.

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